

The Adiabatic Lapse Rate

Thermodynamics is a funny subject. The first time you go through it, you don't understand it at all. The second time you go through it, you think you understand it, except for one or two small points. The third time you go through it, you know you don't understand it, but by that time you are so used to it, it doesn't bother you any more.

Arnold Sommerfeld

Atmospheric thermal profiles, the subject of this essay, are characterized by their gradients or lapse rates. Absent a gravitational field, systems in thermodynamic equilibrium are isothermal with a zero lapse rate. In 1862, Kelvin proposed that the observed decrease in atmospheric temperature with altitude could be explained by an isentropic state induced by earth's gravitational field which he labeled Convective Equilibrium. Shortly thereafter, both Maxwell and Boltzmann showed that velocity distribution functions were not altered by gravity and equilibrium remained isothermal,¹ . . . until the 1960's, when the isentropic model was resurrected by climate scientists, evidently unaware of its short-lived past history. Today, the adiabatic lapse rate remains the cornerstone of global warming calculations with thermal gradients independent of greenhouse gases.

The purpose of this note is to examine in closer detail the thermodynamics of the adiabatic lapse rate. We begin by showing that the isothermal profile is the profile of maximum entropy in the presence of a gravitational field. Our solution starts with an isothermal profile and will show any adiabatic change to another profile results in a decrease of entropy. For this profile,

$$\begin{aligned} T(z) &= T_0 \\ \rho(z) &= \rho_1(z) \end{aligned} \tag{1}$$

and the density profile, $\rho_1(z)$, arbitrary. For adiabatic transformation to an alternative profile, mass and total energy, gravitational plus thermal, must be conserved.

$$\begin{aligned} \text{Mass} &= \int \rho(z) dz \\ \text{Energy} &= g \int \rho(z) z dz + \int \rho(z) dz \int_{T_0}^{T(z)} C_p(T) dT \end{aligned} \tag{2}$$

$C_p(T)$ is the heat capacity at constant pressure per unit mass (Joules/kg/K). This choice allows a system of fixed mass freedom to expand vertically. As a zero reference we choose all mass at T_0 and $z=0$.

For our initial profile,

$$\begin{aligned} E_1 &= g \int \rho_1(z) z dz \\ S_1 &= (g/T_0) \int \rho_1(z) z dz \end{aligned} \tag{3}$$

For our target profile,

$$\begin{aligned} T(z) &= T_2(z) ; T_2(0) = T_0 \\ \rho(z) &= \rho_2(z) \end{aligned} \tag{4}$$

¹ Hyperlinks to these papers are often best found by searches for direct quotations and detailed descriptions are given in Appendix I.

We next find the entropy difference of the new profile by first isothermally changing the density of the isothermal profile, $\rho_1(z)$, to that of the target profile, $\rho_2(z)$. The associated changes in energy and entropy are

$$\begin{aligned}\Delta E_a &= g \int (\rho_2(z) - \rho_1(z)) z dz \\ \Delta S_a &= (g/T_0) \int (\rho_2(z) - \rho_1(z)) z dz = \Delta E_a/T_0\end{aligned}\quad (4)$$

We then adjust the temperature profile to that of the target profile,

$$\begin{aligned}\Delta E_b &= \int \rho_2(z) dz \int_{T_0}^{T(z)} C_p(T) dT \\ \Delta S_b &= \int \rho_2(z) dz \int_{T_0}^{T(z)} \frac{C_p(T) dT}{T}\end{aligned}\quad (5)$$

and then combine these results with the adiabatic constraint

$$\Delta E_a + \Delta E_b = 0 \quad (6)$$

to obtain

$$\Delta S_a + \Delta S_b = - \int \rho_2(z) dz \int_{T_0}^{T(z)} C_p(T) \left[\frac{1}{T_0} - \frac{1}{T} \right] dT \leq 0 \quad (7)$$

Any adiabatic deviation from the isothermal profile reduces total entropy and the isothermal profile is therefore that of maximum entropy.

Total entropy for the perturbed profile is

$$S_2 = S_1 + \Delta S_a + \Delta S_b \quad (8)$$

From Eqs 4, 5 and 6

$$\begin{aligned}\frac{d S_1}{d z} &= g \rho_1 z / T_0 \\ \frac{d \Delta S_a}{d z} &= (g/T_0)(\rho_2(z) - \rho_1(z)) z \\ \frac{d \Delta S_b}{d z} &= C_p \rho_2(z) \ln(T(z)/T_0)\end{aligned}\quad (9)$$

hence

$$d S_2 / d z = \rho_2(z) [g z / T_0 + C_p \ln(T(z)/T_0)] \quad (10)$$

This derivative is important, for positive values are a necessary condition for the absence of convection.² The isentropic or adiabatic profile is therefore

$$T(z) = T_0 e^{-g z / C_p T_0} \quad (11)$$

This is one member of a set of exponential lapse-rate functions ,

$$T(z) = T_0 e^{-\kappa z / T_0} \quad (12)$$

The characteristic distance describing these profiles, T_0/κ , is of order 40 km, and linear expansion of the exponential, $T(z) \simeq T_0 - \kappa z$, is appropriate within the troposphere.

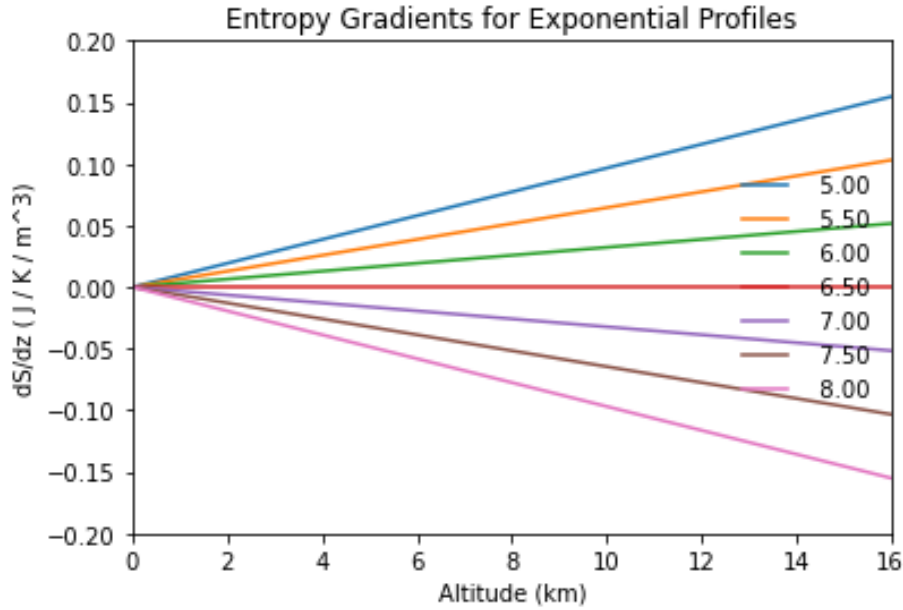
² Fluid Dynamics, "§4. The condition that convection is absent", L.D. Landau and E.M. Lifshitz, Addison-Wesley (1959)

To evaluate Eq. 10, we note

$$\begin{aligned} \rho(z) &= P(z)/RT(z) \\ \ln(P(z)/P_0) &= -(g/R\kappa)(e^{\kappa z/T_0} - 1) \end{aligned} \quad (13)$$

Given the parameters

$g = 9.80665$	m/s^2
$R = 287.06$	$J/kg/K$
$C_p = 1509$	$J/kg/K$ ($g / C_p = 6.5 K/km$)
$P_0 = 102890.0$	$Pa = J/m^3$
$T_0 = 285.0$	K

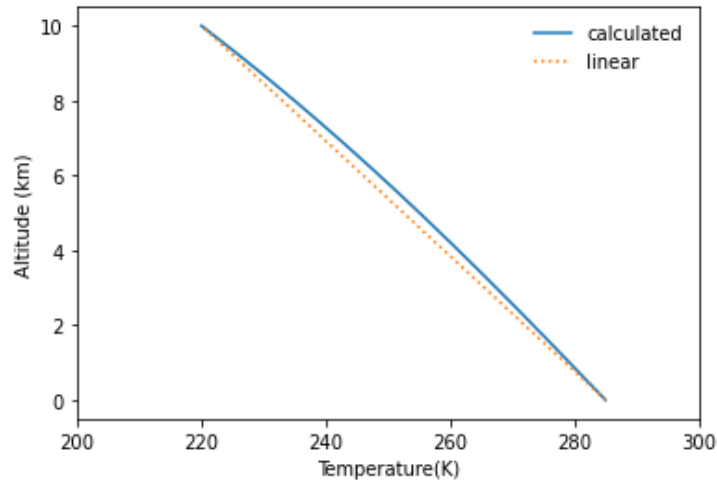


At the chosen adiabatic lapse rate, 6.5 K/km, entropy density gradients are zero but they are also zero for all constant lapse rate profiles at the surface, with linearity implying an entropy quadratically varying with altitude. The 2nd Law defines the relationship between entropy formation, energy flux and temperature,

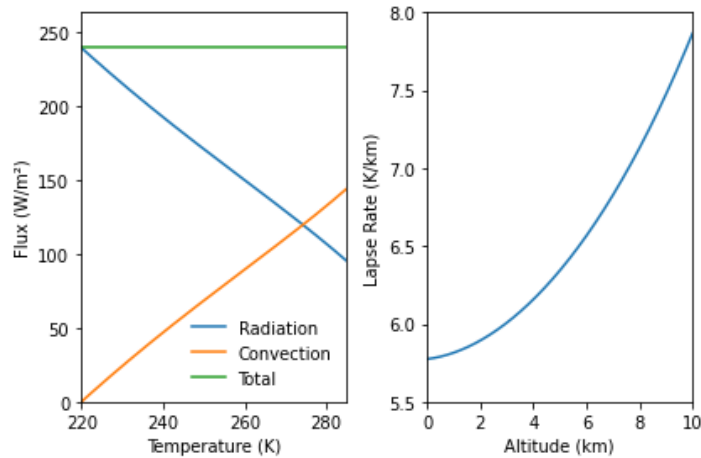
$$\frac{dS}{dt} = \mathbf{J}_U \cdot \nabla(1/T) = \frac{dS}{dz} \frac{dz}{dt} \quad (14)$$

In the region $dS/dz > 0$, energy flux will be towards cooler temperatures, vanishing as dS/dz approaches zero. At the adiabatic lapse rate, the energy flux becomes zero. Textbooks frequently offer a 'parcel' rationalization for finite flux within an isentropic system. Should we adiabatically raise a thermally insulated parcel, e.g. a balloon, slowly enough to avoid viscous dissipation, no net work is required. But, the volume previously occupied by the parcel will then be filled by the surrounding isentropic fluid leaving the system and its center of gravity unchanged. No climatologist today, no matter their opinions on GHG issues, questions the role of the adiabatic lapse rate. It is the third rail of climatology and the only sanctioned rationale for thermal gradients.

Some might well argue that the adiabatic lapse rate has been experimentally confirmed by radiosonde measurements. Consider the following plot:



It certainly looks like a 6.5 K/km lapse rate should be a good enough approximation. But, other properties for this same calculation beg to differ.



These plots were obtained with a model based on prescribed boundary values for temperature and fluxes (*vide infra*).

Properties of especial interest are differential in character. We shall define *Thermal Sensitivity* as the ratios of changes in total flux due to small changes in boundary temperatures. An unresolved subject of substantial interest is the differential relation between energy flux and temperature,

$$\delta J_U = \left(\frac{\partial J_U}{\partial T_1} \right)_{T_2} \delta T_1 + \left(\frac{\partial J_U}{\partial T_2} \right)_{T_1} \delta T_2 \quad (15)$$

with T_1 the surface temperature and T_2 the temperature at the top of the troposphere. We distinguish two cases, Case I: $\delta T_2 = 0$ and Case II: $\delta T_2 = \delta T_1$. In the former, boundary temperatures are independent variables, in the latter they are coupled by an unspecified constraint such as a preordained lapse rate. It proves an interesting experiment to explore a set of eight profiles having identical boundary values for flux and temperature, but capricious variations of radiative and convective parameters betwixt.

Sensitivities were calculated for 240 W/m^2 total energy fluxes between 285K and 220K boundaries and 0.01K changes for calculation of partial derivatives:

Case I: $4.39 \pm 0.21 \text{ Watts/m}^2/\text{K}$

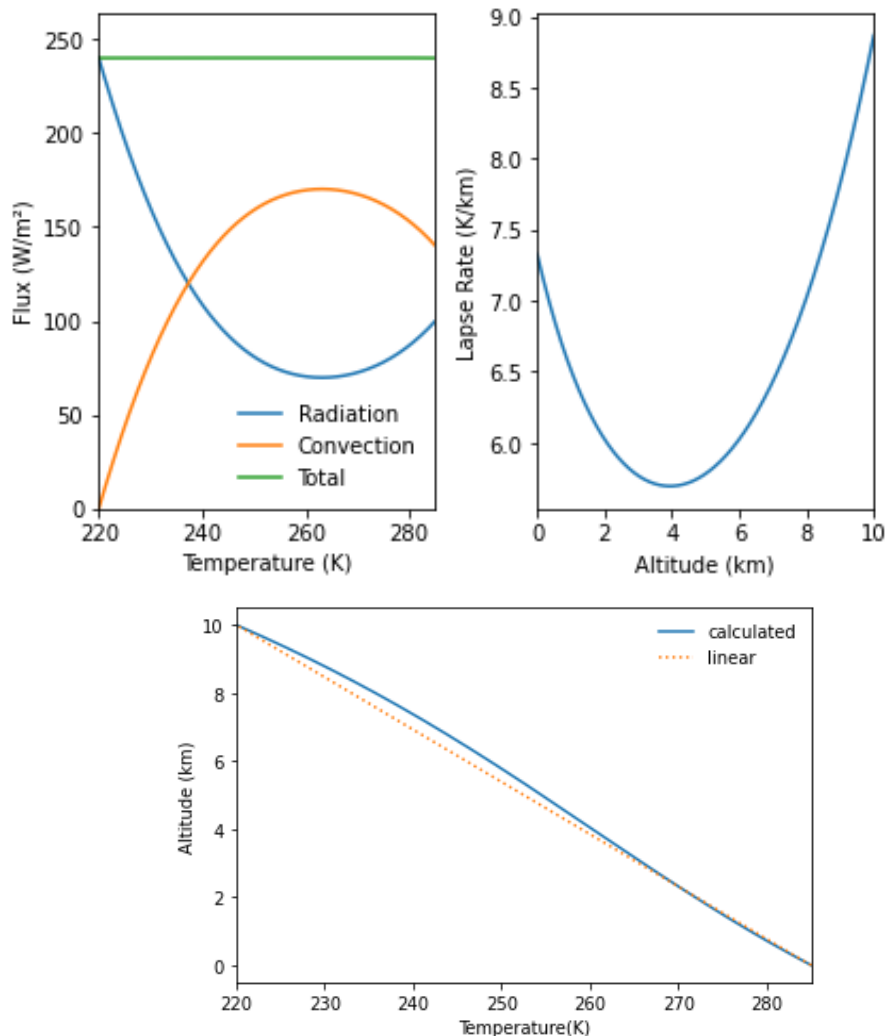
Case II: $1.33 \pm 0.38 \text{ Watts/m}^2/\text{K}$

The T_1 partial is always positive, the T_2 negative, hence the reduced sensitivity for Case II. The conventional definition of climate sensitivity is the surface temperature change for a CO_2 doubling of 3.7 W/m^2 ,

Case I: $0.84\text{K} \pm 0.04\text{K}$

Case II: $2.78\text{K} \pm 0.79\text{K}$

Case I describes models for which thermal gradients are influenced by greenhouse gases. Case II describes models for which thermal gradients are independent of CO_2 , e.g. the adiabatic lapse rate. Currently, the majority of climatologists will assert increases beyond 1.5K constitute an irreversible, existential threat for mankind. What's most surprising about Case I, however, is not that the warming lies well outside doomsday scenarios, but the insensitivity to internal details, implying global warming is primarily a boundary value problem. One of the more extreme profiles explored was strongly biased towards convection in the lower troposphere, lapse rates varying dramatically, yet the plot of altitude vs. temperature appears close enough to linear to elude radiosonde detection. (Case I Sensitivity: $4.08 \text{ W/m}^2/\text{K}$)



When these experiments were repeated for the set of eight profiles with boundary fluxes doubled, other parameters unchanged,

Case I: 8.47 ± 0.48 Watts/m²/K

Case II: 1.96 ± 0.68 Watts/m²/K

Case I results argue a sensitivity proportional to total energy flux density with a proportionality coefficient of 55-57K, not dissimilar to the 65K temperature differential assumed. As a test, eight calculations for a linear model varying T_1 from 265 to 300K, total flux fixed at 240 Watts/m² yielded proportionality coefficients of 53.3 ± 2.0 K. Empirically,

$$T_1 \text{ Sensitivity} = \gamma J_{total} / (T_1 - T_2) \quad (16)$$

with the dimensionless parameter $\gamma = 1.22 \pm 0.04$.

It should be emphasized this result is not based on physical observations but a summary of numerous solutions to a non-trivial mathematical model. The HBC model begins with given boundary values for temperature and both radiative and convective flux.³ Temperature is then described internally as a polynomial of altitude, typically with seven variable coefficients. Internally, fluxes for radiation and convection are defined as the product of the temperature's gradient and hypothetical polynomial functions of altitude matching boundary values. Typically, these functions may exaggerate variations well beyond rational physical expectation. The temperature coefficients are found by minimizing deviations of the calculated total flux from its mean value and sensitivities from changes in total flux to 0.01K changes in boundary temperatures. Despite internal arbitrariness, Eq. 16 suggests sensitivity can be estimated to within a few percent wholly from boundary values for temperature and flux.

The current obsession over carbon dioxide boils down to the ratio of change in thermal energy exiting the earth's atmosphere to a perturbation in its surface temperature. Temperature is a funny property. It is not a physical property expressible in units of mass, distance and time, but defined by the 2nd Law, jointly with another funny property, entropy, in terms of a true physical property, energy.

$$\nabla \cdot \mathbf{J}_s(r) = \mathbf{J}_u(r) \cdot \nabla \left(\frac{1}{T(r)} \right) \quad (17)$$

What makes this definition unique is that entropy is required to be an exact differential and thermodynamic states thus independent of the paths by which they have evolved. Thermodynamic states have no memory of their origins, that information having been subsumed as entropy. As a benefit, we need not specify 10^{24} initial parameters to obtain reproducible results for a mole of material. As a detriment, we have no handle for defining temperature in time-dependent situations.

We began this note showing total entropy for an isothermal system to be a maximum, independent of gravitational fields. This is an integral property. In non-equilibrium systems, models of local equilibria are often invoked for differential volume elements. Such approximations fail, however, should gradient properties be relevant. Equation 10 describes the local entropy of a volume element and reflects the collected wisdom of the climate science community in its Radiative-Convective-Equilibrium models. These models typically lead to Case II sensitivities requiring draconian countermeasures, essentially because the adiabatic lapse rate assumption prescribes perturbations for disparate surfaces.

3 pdq2021.000webhostapp.com/HBC_Model.pdf

Dissipation is another funny property. It is the surface integral of $\nabla \cdot J_F$, the divergence of the flux of free energy, and entirely a boundary value function. For thermodynamic steady states

$$Dissipation = J_U \frac{T_1 - T_2}{T_1} \quad (18)$$

is the constant work required to forestall relaxation towards isothermal equilibrium. For the troposphere, 27% of a transiting 240 Watts/m^2 is continuously being dissipated. Nevertheless, it seems inevitably assumed that the Boltzmann distribution, rigorous only for isothermal systems, applies throughout. Dissipation is a minimum for thermodynamic steady states and a minimum for J_U when boundary temperatures are fixed. This is illustrated by changing the number of variable coefficients in $T(z)$.

Coefficients	Mean Flux	Std. Dev.
1	240.679669	9.529233
2	239.430446	0.460400
3	239.688475	0.106538
4	239.582352	0.012007
5	239.556492	0.001852
6	239.547983	0.000480
7	239.542772	0.000083
8	239.540624	0.000017
9	239.539490	0.000004
10	239.539703	0.000005

If J_U is fixed by solar output, thermodynamics will minimize $T_1 - T_2$. Given the choice for removing energy from the surface should nature favor a horizontal or a vertical path? Clouds are a current bugaboo for the climate modeler. Do they inhibit radiation or facilitate convection? Might they be implicit among the Case I models analyzed?

How can we assert CO_2 doubling will raise surface temperatures 0.8K when RCE models have difficulty establishing where the decimal point is? The latter assert, without justification, the adiabatic profile to be an equilibrium property. Unconstrained, a system would then relax to this profile. Both Maxwell and Boltzmann, over a century ago, disproved this notion and Maxwell even described how to construct a perpetual motion device from two columns of different gases were it otherwise. RCE models (Case II) create a thermal discontinuity at the tropopause without considering its aftereffects on the stratosphere. Equilibrium systems do not dissipate energy, the troposphere does. The adiabatic lapse rate defines a necessary condition for the absence of convection. Its role after a system has entered a convective state is unclear. The greatest difficulty with RCE models, however, is finding a non-zero value for convective flux. It is not a function of the lapse rate for which it is assumed responsible. Case I models are the clear favorite with respect to dissipation. They assume first-order perturbations of tropospheric boundary temperatures are independent with thermal gradients determined by the dissipation of both radiative and convective fluxes.

If nothing more, our discussion should demonstrate a distinction between physical and thermodynamic models. Temperature and entropy play no roles in kinetic models for the solar system or the carbon dioxide molecule. They are, however, the *sine qua non* for models describing thermodynamic states.

APPENDIX I

At the January 1862 meeting of the Manchester Literary and Philosophical Society, a paper, On the Convective Equilibrium of Temperature in the Atmosphere, by Professor Wm. Thomson was read by Dr. Joule: *“When all the parts of a fluid are freely interchanged and not subject to the influence of radiation and conduction, the temperature of the fluid is said by the Author to be in a state of convective equilibrium.”*

In May 1866, J.C. Maxwell responds: On the Dynamical Theory of Gases, Philosophical Transactions of the Royal Society of London, Vol. 157, p. 86 (1867), *“The left-hand side of equation (147), as sent to the Royal Society, contained a term, the result of which was to indicate that a column of air, when itself, would assume a temperature varying with the height, and greater above than below. The mistake arose from an error in equation (143). Equation (147), as now corrected, shows that the flow of heat depends on the variation of temperature only, and not on the direction of the variation of pressure. A vertical column would therefore, when in thermal equilibrium, have the same temperature throughout.”*

In October 1875, Ludwig Boltzmann writes: Über das Wärmegleichgewicht von Gasen, auf welche äußere Kräfte wirken, Sitzungsberichte der Mathematisch-Naturwissenschaftlichen, Vol. 72-II, p. 443 (1876), *“Aus dieser Formel folgt, daß trotz der Wirksamkeit der äußeren Kräfte für die Richtung der Geschwindigkeit irgend eines der Moleküle jede Richtung im Raume gleich wahrscheinlich ist, ferner dass in jedem Raumelemente des Gases die schwindigkeitsvertheilung des Gases genau ebenso beschaffen ist, wie in einem Gase von gleicher Temperatur, auf das keine Aussenkräfte wirken. Der Effect der äusseren Kräfte besteht blos darin, dass sich die Dichte im Gase von Stelle zu Stelle verändert und zwar in einer Weise, welche schon aus der Hydrostatik bekannt ist.”*

Google's translation: *“From this formula, it follows that in spite of the effectiveness of the external forces for the direction of the velocity of any of the molecules, each direction in space is equally probable, furthermore that in each space element of the gas the velocity distribution of the gas is exactly the same as in a gas of same temperature, on which no external forces act. The effect of the external forces consists merely in the fact that the density in the gas changes from place to place in a manner which is already known from hydrostatics.”*